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SURFACE INTEGRALS OF POINCARE ALGEBRA IN ASHTEKAR'S FORMALISM

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ABSTRACT. *When an asymptotically flat spacetime is considered the new canonical formulation of General Relativity invented by Ashtekar requires a due account of surface integrals which are necessary to realize the Poincare algebra. In particular, the Poisson brackets should be redefined to get reasonable results.*

The Hamiltonian formalism proposed by Ashtekar (1986), which uses as field variables spinor soldering forms or triads instead of a 3-metric, is of considerable interest for the quantum gravitational theory. Jacobson and Smolin (1988), Rovelli and Smolin (1990) have already developed these applications. However, until now not enough attention has been paid to quantization of the asymptotically flat spacetime. In such a problem an infinite group of diffeomorphisms which preserve boundary conditions at spatial infinity is of great importance. The Poincare group can be factored out from it. In the ordinary canonical formalism, the Poincare group generators are expressed in terms of surface integrals taken over a two-dimensional sphere at spatial infinity, as was shown for the first time by Regge & Teitelboim (1974).

The two-fold nature of these surface integrals was disclosed (Regge & Teitelboim, 1974): they guarantee differentiability of the Hamiltonian at the prescribed boundary conditions and they also enter the Poisson brackets algebra

$$\{H(N^{\alpha}),\ H(M^{\beta})\} \simeq H(L^{\gamma}), \tag{1}$$

where

$$L^{\gamma} = [N,M]^{\gamma}, \tag{2}$$

and every vector field represents coordinate transformation from the asymptotic Poincare group. The second aspect was discussed much less than the first one. But the straightforward calculation of all the surface terms in the Poisson brackets, in the general case first done by Soloviev (1985), gives us many advantages, for example, a possibility of search for new boundary conditions or evaluation of central charges. Later this method was mentioned by Brown & Henneaux (1986a) with a remark that "such a calculation is typically very cumbersome". The same authors proved in other publication (Brown & Henneaux, 1986b) that a Poisson bracket of two differentiable in Regge-Teitelboim's sense generators is also a differentiable generator.

Here we will show how the Poincare group manifests itself in Ashtekar's canonical formalism where the boundary conditions for new variables are taken from Ashtekar (1987) and Ashtekar et al. (1987). As the basic variable we choose the triad related to the 3-metric as follows:

$$\gamma\gamma^{ij} = \tilde{E}^{ia}\,\tilde{E}^{ja}, \tag{3}$$

(in our notations, the triad indices are: a,b,c,...; the spatial ones are: i,j,k,...). The Hamiltonian contains three additional constraints as compared to the standard General Relativity:

$$H = \int\left[\underset{\sim}{N}\,\tilde{E}^{ia}\,\tilde{E}^{jb}\,F_{ij}^{\;c}\,\epsilon^{abc} + 2i\,N^{i}\tilde{E}^{ja}\,F_{ij}^{\;a} + \hat{N}^{a}D_{i}\,\tilde{E}^{ia}\right]d^{3}x. \tag{4}$$

These new constraints generate the triad rotations and are analogous to the Gaussian constraints of the gauge theories. Conjugate to $\tilde{E}^{ia}$ is the connection A_i^a:

$$\{\tilde{E}^{ia}(x), A_j^b(y)\} = \frac{i}{2}\,\delta^i_j\,\delta^{ab}\delta(x,y). \tag{5}$$

The constraint algebra was obtained by Ashtekar(1987) and Ashtekar et al. (1987), where, as usual, the surface integrals arising in calculations of the Poisson brackets by the formula

$$\{H_1,H_2\} = \frac{i}{2}\int\left[\frac{\delta H_1}{\delta\tilde{E}^{ia}}\frac{\delta H_2}{\delta A_i^a} - (1\leftrightarrow 2)\right]d^3x, \tag{6}$$

were not taken into account.

Following our approach proposed by Soloviev (1985), we find the Poisson brackets between the constraints not discarding any surface terms. The appearance of these surface integrals is not very transparent (Soloviev, 1991) and we do not display them here. According to Regge-Teitelboim's ideology and to Brown-Henneaux's theorem (Brown

& Henneaux, 1986b) we can expect these surface terms to coincide with those obtained through the Hamiltonian variation at Ashtekar's boundary conditions (Ashtekar, 1987; Ashtekar et al., 1987):

$$H\,(N^{\alpha}) \approx \oint 2\underset{\sim}{N}\, \epsilon^{abc}\, \tilde{E}^{ia}\, \tilde{E}^{jb}\, A^{c}_{j}\, dS_{i} + 2i \oint (N^{i}\, \tilde{E}^{ja} - N^{j}\, \tilde{E}^{ia}) A^{a}_{j}\, dS_{i}, \tag{7}$$

but unfortunately the results are different. For example, for the exact Schwarzschild solution (Bengtsson, 1990),

$$\tilde{E}^{ia} = (1-k/r)^{-1/2}\delta_{ia} + [1-(1-k/r)^{-1/2}]n_i n_a, \tag{8}$$

$$A^{a}_{i} = 1/r[(1-k/r)^{1/2}-1]\epsilon_{iab}n_b, \tag{9}$$

where $n_i = x^i/r$, by commutating the spatial translation $N^i=\lambda^i$ and the rotation $M^j=\bar{\omega}_{jk}x^k$, instead of zero momentum we have

$$\{H(N^i),H(M^j)\} = i\,\frac{4\pi k}{3}\,\epsilon_{ijk}\lambda_i\bar{\omega}_{jk}, \tag{10}$$

and by commutating the same spatial translation and the boost $\underset{\sim}{M}=\bar{B}_k x^k$, instead of the Schwarzschild mass $8\pi k$ multiplied by the time translation $\lambda^i\bar{B}_i$ we have

$$\{H(N^i),H(\underset{\sim}{M})\} = \frac{8\pi k}{3}\,\lambda^i\bar{B}_i. \tag{11}$$

The reason for these unpleasant results, as we have found (Soloviev, 1991), is that the Ashtekar's transformation (Ashtekar, 1986; 1987; Ashtekar et al., 1987) is not strictly canonical. When surface integrals are taken into account the Poisson bracket

$$\left\{\int N^{ia}(x)A^{a}_{i}(x)d^3x,\ \int M^{jb}(y)A^{b}_{j}(y)d^3y\right\} \tag{12}$$

is not zero and the correct Poisson bracket for Hamiltonian generators should be

$$\{H_1,H_2\}=\{H_1,H_2\}_{old}+\{H_1,H_2\}_{correction}, \tag{13}$$

where

$$\{H_1,H_2\}_{correction}=\int d^3x \int d^3y\, \frac{\delta H_1}{\delta A^{a}_{i}(x)}\,\frac{\delta H_2}{\delta A^{b}_{j}(y)}\,\{A^{a}_{i}(x),A^{b}_{j}(y)\}=$$

$$= \frac{i}{4}\oint \frac{dS_k}{E}\,\epsilon^{acd}(\delta^{k}_{j}\delta^{bd}E_{ic}-E_{ib}E_{jc}E^{kd})\left[\frac{\delta H_1}{\delta A^{a}_{i}}\,\frac{\delta H_2}{\delta A^{b}_{j}} - (1\leftrightarrow 2)\right]. \tag{14}$$

With this redefinition of the Poisson brackets the algebra of Poincare group generators can be definitely realized for the Schwarzschild solution and the generators have values that coincide with the Ashtekar's expressions.

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